Intersection of triadic Cantor sets with their translates. II. Hausdorff measure spectrum function and its introduction for the classification of Cantor sets

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Abstract

Initiated by the purpose of classification of sets having the same fractal dimension, we continue, in this second paper of a series of two, our investigation of intersection of triadic Cantor sets and their use in the classification of fractal sets. We exploit the infinite tree structure of translation elements to give the exact expressions of these elements. We generalize this result to a family of uniform Cantor sets for which we also give the Hausdorff measure spectrum function (HMSF). We develop three algorithms for the construction of HMSF of triadic Cantor sets. Then, we introduce a new method based on HMSF as a way for tracing the geometrical organization of a fractal set. The HMSF does carry a huge amount of information about the set to likely be explored in a chosen way. To extract this information, we develop a one by one step method and apply it to typical fractal sets. This results in a complete identification of fractals.

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1. Introduction

As raised by Mandelbrot himself, the basic ideas of fractal geometry have been viewed, after the first maturation stage of the field, as being astonishingly simple [1]. This view no doubt is due to easy intuition, picturing and applications, the latter being sometimes abusive and oversimplified in many senses. A second look reveals, however, many unexplored facets to get deepen and likely be translated in useful tools to be added in the fractal toolbox. The most used and popularized concept in fractal geometry was the fractal dimension. Only in the last decade, additional tools to get rid of the degeneracy character of fractal dimension have been developed. Among them, few have been devoted to fractal features of ‘texture’, a broad concept called lacunarity by Mandelbrot [2]. Lacunarity has been introduced qualitatively, as a notion of texture based on the tendency of a set to get gaps, a concept strongly related to the shapes of those gaps as well as to their distribution. Originally, fractals which motivated the paradigm of lacunarity satisfy the particular property of discrete scale invariance (DSI)—a weaker kind of (continuous) scale invariance symmetry—in the sense that they are scale invariant only under a discrete set of dilatations. Those structures are common in the physical world and the Cantor-like sets are the simplest canonical examples. In several papers, Sornette relates the importance and relevance of DSI to physics and listed some physical mechanisms that can lead to it [3,4]. This author makes use of complex dimensions to study DSI. DSI property is mainly caused by a break of symmetry of diverse origins.

The gliding-box algorithm (GBA), introduced in a lacunarity study context, has been derived from the box-counting method (BCM) by gliding a box over the set, one unit at a time in a discrete manner [12]. As we know, BCM (hence
2. Theoretical determination of the Hausdorff measure spectrum function of the intersection of the triadic Cantor set with itself

Let $C$ be the triadic Cantor set generated from the unit interval $[0, 1]$, and consider the intersection of $C$ with its translation $C + t$ that we denote with $I(t) = C \cap (C + t)$. Instead of considering only positive shift numbers $t : 0 \leq t \leq 1$ as we did in [11], we will work in this section on the total significant range of $t : -1 \leq t \leq 1$.

In [11], we proved that the Hausdorff measure of $I(t)$ can only have values from a discrete set $M = \{0\} \cup \{1/2^i\}_{i=0}^{\infty}$ when $t$ varies between $-1$ and $1$ at dimension $s = \log 2 / \log 3$.

We keep the same notation as in [11] so we denote by $T_t$ the translation elements corresponding to the discrete spectrum, i.e., $T_t = \{ t : \mathcal{H}^s(I(t)) = 1/2 \}$. A natural question was to determine the exact forms of those elements. In [11], we have focused on the construction of $T_t$ and their inter-relationship. In this paper, we succeed in giving the exact expressions of these translation elements. Indeed, we prove that there exists an infinite tree structure between $T_t$, where the number of branches to any knot of the tree is infinite.

**Theorem 1.** Let $C$ be the triadic Cantor set and let $T_t = \{ t : \mathcal{H}^s(I(t)) = 1/2 \}$, where $s = \log 2 / \log 3$. Then $T_t$ can be organized into an infinite tree where the number of branches to any knot of the tree is infinite. They are given by:

$$T_n = \bigcup_{1 \leq a_1, a_2, \ldots, a_n < \infty \atop a_1, \ldots, a_n \in \{-1, 1\}} \left\{ (-1)^{\alpha_1} \frac{2}{3^{a_1}} + (-1)^{\alpha_2} \frac{2}{3^{a_1+a_2}} + \cdots + (-1)^{\alpha_n} \frac{2}{3^{a_1+a_2+\cdots+a_n}} \right\}. $$
Proof. The infinite tree can be defined as follows:

We choose the zero translation as the root of the infinite tree as $T_0 = \{0\}$.

Since we have proved that only at the shifts $t = \pm 2/3^n$, $i = 1, 2, \ldots, I(t)$ can have its Hausdorff measure 1/2 at dimension $s$, we write $T_t = \bigcup_{s = 0}^{2^{n-1}} \{\rho_t{t_1}^s \}$ where $t' = (-1)^2 2/3^n$, $i = 1, 2, \ldots, \sigma = -1$ corresponds to the left shifts and $\sigma = 1$ for the right ones. As the infinite tree will be generated in the same way as from $T_0$ to $T_1$ across the generations, we simply note $T_1$ as $T$, and call $T$ the generator of the infinite tree.

Assume that $T_n = \bigcup_{s = 0}^{2^n} \{\rho_t{t_1}^s \}$ is obtained.

We can prove that

$$T_{n+1} = \bigcup_{s = 0}^{2^{n+1}} \{\rho_t{t_1}^s \}$$

where

$$\rho_t{t_1}^s \frac{\rho_t{t_1}^s + \frac{1}{3^{t_1} + 2^{n+1} + i_k}}{3^{t_1} + 2^{n+1} + i_k}.$$ 

We can see it in this way:

From the possible structure of the intersection $C \cap (C + t)$ that we found in [11], corresponding to the shift $t = \rho_t{t_1}^s \in T_n$, $C \cap (C + t)$ should be composed of $2^{t_1} + t_2 + \ldots + t_n$ small triadic Cantor sets of length $1/3^{t_1} + 2^{n+1}$ which measure combines to a Hausdorff measure of $1/2^n$. Also, any Hausdorff measure $1/2^n$ can and only can be produced from a certain shift $t = \rho_t{t_1}^s \in T_n$ by a relative shift of length $\pm 2/3^{n+1}$ within the scale $1/3^{t_1} + 2^{n+1}$. As the shift $2/3^{n+1}$ or $-2/3^{n+1}$ to a triadic Cantor set produces the intersection composed of $2^{n+1}$ small triadic Cantor sets of relative length $1/3^{n+1}$, we find with this shift that the intersection of Cantor set with its translation will be composed of $2^{t_1} + t_2 + \ldots + t_n$ + $2^{n+1} = 2^{t_1} + t_2 + \ldots + t_n + 1$ small triadic Cantor sets of length $1/3^{t_1} + 2^{n+1} + i_k$. This measure combines to a Hausdorff measure of $1/2^{n+1}$.

Like what we did from $T_0$ to $T_1$, the operation that we performed from $T_n$ onto $T_{n+1}$ is that exactly, for every $t = \rho_t{t_1}^s \in T_n$, to carry out a group shift $T$ to $t$ within the scale $1/3^{t_1} + 2^{n+1} + i_k$. Clearly, we have

$$T_{n+1} = \bigcup_{s = 0}^{2^{n+1}} \{\rho_t{t_1}^s \}$$

The right terms in the union notation are one-by-one disjoint as any $t = \rho_t{t_1}^s \$ is discarded at least a distance $1/3^{t_1} + 2^{n+1} + i_k$ from each other by the structure of $T$.

Simply, we write

$$T_{n+1} = \bigcup_{s = 0}^{2^{n+1}} \{\rho_t{t_1}^s \}$$

where

$$\rho_t{t_1}^s \frac{\rho_t{t_1}^s + \frac{1}{3^{t_1} + 2^{n+1} + i_k}}{3^{t_1} + 2^{n+1} + i_k}$$

as

$$T = \bigcup_{s = 0}^{2^{n+1}} \{\rho_t{t_1}^s \}.$$ 

Clearly, if we consider each element $\rho_t{t_1}^s \in T_n$ as a knot of $n$th generation of the infinite tree, $\rho_t{t_1}^s + T/3^{t_1} + 2^{n+1} + i_k$ are branches belonging all to $T_{n+1}$.

Recursively

$$\rho_t{t_1}^s = \frac{\rho_t{t_1}^s}{3^{t_1}} + \ldots + \frac{\rho_t}{3^{t_1} + 2^{n+1} + i_k} = (-1)^{t_1} \frac{2}{3^{t_1}} + (-1)^{t_2} \frac{2}{3^{t_1} + 2^{n+1} + i_k}. $$


Finally, we get an exact expression for $T_n$:

$$T_n = \bigcup_{1 \leq i, j \leq n-1, \ i + j \neq n} \left\{ (-1)^{\xi_i} \frac{2}{3^n} + (-1)^{\xi_j} \frac{2}{3^n} + \cdots + (-1)^{\xi_n} \frac{2}{3^n} \right\}. \quad \square$$

3. The spectrum of a family of uniform Cantor sets

Now, in order to test our classification method, we need a sufficiently large family of sets having the same fractal dimension. Some of these sets will have, at a first glance, a different structure but, as we will show in the section below, are similar (up to a scale), which is also reflected through our method.

Let us extend our knowledge of the exact discrete spectrum and the associate translation elements for the triadic Cantor set to some uniform Cantor sets.

For this purpose, we define a family of iterative function systems (IFS), that we call a ‘comb’ IFS which will be used as a representative for the class of IFS that give rise to comb Cantor sets (since within each of their different parts, organized from left to right, mass and voids are of the same length, with a possible handle at the right end). It’s clear that a Cantor set can be defined by several IFS.

First, let us give the definition and construction rule of comb Cantor sets. Without loss of generality, we use the unit interval $I = [0, 1]$ as the initiator. A Cantor set $F$ belongs to this family if it can be generated in the following way:

**Definition 1.** If $m$, $n$ are two positive integers satisfying $m \geq 2n - 1$, and if $\{S_i\}_{i=1}^n$ is an IFS given by

$$S_i : [0, 1] \rightarrow [0, 1] \quad \text{by} \quad x \mapsto \frac{1}{m} x + \frac{2(i-1)}{m},$$

$i = 1, \ldots, n$, then we call the attractor $F$ of this IFS an $(m, n)$ comb Cantor set and denote it by $F(m, n)$.

Having defined the comb sets through their dynamic generation, let us give their geometric construction and some terminologies which will be useful in the following.

The first step of the construction is to divide $I$ into $m$ equal subintervals, $[(i-1)/m, i/m]$, $i = 1, \ldots, m$, and to choose $n$ subsequent subintervals on the odd positions from left to right as $F_i = [(2(i-1))/m, (2i-1)/m]$, $i = 1, \ldots, n$; write the union as $F^1 = \bigcup_{i=1}^n F_i$.

The next step is to perform, on each $F_i = [(2(i-1))/m, (2i-1)/m]$, $i = 1, \ldots, n$, the same operation as done in the first step, i.e., divide each (sub)interval into $m$ equal subintervals, and choose the subsequent subintervals on the odd positions from left to right to get subintervals $F_{ij}$, $i = 1, \ldots, n$, $F_{ij} \subset F_i$; each one is of length $1/m^2$; write $F^2 = \bigcup_{j=1}^m F_{ij}$.

In the same fashion, at the $l$th step, we get $\{F_{i_1i_2\ldots i_l}\}_{i_1, i_2, \ldots, i_l=1}^n$, each one is of length $1/m^l$ and we write $F^l = \bigcup_{j=1}^m F_{i_1j_2\ldots i_l}$. $F = \bigcap_{l=1}^\infty F^l$ is an $(m, n)$ comb fractal in the sense we defined above.

We call $F \cap F_i$ the $i$th finger of the first generation; $F \cap F_{ij}$ the $ij$th finger of the second generation; $F \cap F_{i_1i_2\ldots i_l}$ the $i_1i_2\ldots i_l$th finger of the $l$th generation, etc. Also $1/m^l$ is the unit gap in the $l$th generation.

In the following, we calculate the Hausdorff spectrum of the comb Cantor set and establish its discreteness.

**Spectrum calculation:** In this part, we consider $F \cap (F + t)$ only when $t$ is a multiple of $1/m^l$ for a certain $l$, i.e., $t$ has a finite $m$-ary expansion. By symmetry, we only take $t > 0$.

Reversibly, we prove that the nontrivial Hausdorff measure spectrum can only take a value from the form $(1/n)^l(2/n)^l \cdots ((n-1)/n)^{k-1}$, where $i_1i_2i_3\ldots i_{n-1} \in \mathbb{Z}^+$. It is clear that, when $t$ is a multiple of $1/m^l$, the Hausdorff measure of $F \cap (F + t)$ can only take a value of $1/n, 2/n, \ldots, (n-1)/n, 1$.

Suppose that, when $t$ is a multiple of $1/m^l$, the assumption is true.

Then if $t$ is a multiple of $1/m^l$, we can write $t = (p/m^{l-1}) + (k/m^l)$, $p$ is a positive integer and $k$ is an integer between 0 and $m - 1$.

By the assumption, $F \cap (F + (p/m^{l-1}))$ has Hausdorff measure of the form $(1/n)^l(2/n)^l \cdots ((n-1)/n)^{k-1}$, and is composed of $2^l \cdots (n-1)^{k-1}m^{l-1-n/2}i_{i-1}$ fingers of $(l-1)$th generation.

If we apply a right translation $k/m^l$ to $F + (p/m^{l-1})$, the same operation will be done at each of the $(l-1)$th generation finger of $F \cap (F + (p/m^{l-1}))$. 

As the shift is a multiple of $1/m^l$, each $(l - 1)$th generation finger will leave only a number of $l$th generation fingers that is a value from $1$ to $n$ that we can denote by $k$.

Then $F \cap (F + t)$ is composed of $k \cdot 2^{n_{l+1}} \ldots (n-1)^{l_{n_{l+1}}}$ fingers in each generation, the Hausdorff measure should be $(1/n)^{l_1}(2/n)^{l_2} \ldots (k/n)^{l_{n_{l+1}}} \ldots ((n-1)/n)^{l_{n_{l+1}}}$.

The discreteness of the Hausdorff measure spectrum: If $t$ is not a multiple of $1/m^l$, i.e., $t$ does not have a finite $m$-ary expansion, then $\mathscr{H}^s(F \cap (F + t)) = 0$.

Suppose $t = \sum_{i=1}^{\infty} k_i/m^l$ where $0 \leq k_i < m$. When $t$ has an equivalent finite $m$-ary expansion, this case will reduce to the above one. Hence, we only consider the case when $t$ cannot have a finite $m$-ary expansion.

As we know, for a finite $m$-ary expansion shift given by $\sum_{i=1}^{l} k_i/m^l$, we have:

$$\mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l\right)\right) = \left(\frac{1}{n}\right)^{i_1} \ldots \left(\frac{n-1}{n}\right)^{i_{n_{l+1}}} \text{ where } i_1, i_2, \ldots, i_{n_{l+1}} \in \mathbb{Z}^+.$$

On the other hand,

$$\mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{\infty} k_i/m^l\right)\right) \leq \max \left(\mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l\right)\right), \mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l + k_{l+1}/m^{l+1}\right)\right)\right).$$

One term in the max is zero. Without loss of generality, we suppose

$$\mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l\right)\right) \neq 0.$$ 

Then

$$\mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l+1} k_i/m^l\right)\right) = \mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l\right)\right) \cdot \frac{p_{l+1}}{n} \text{ where } 0 \leq p_{l+1} \leq n.$$ 

Iteratively, for any $N > 0$, we have

$$\mathscr{H}^s(F \cap (F + t)) \leq \mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l+N} k_i/m^l\right)\right) = \mathscr{H}^s\left(F \cap \left(F + \sum_{i=1}^{l} k_i/m^l\right)\right) \cdot \frac{p_{l+1} p_{l+2} \ldots p_{l+N}}{n}$$

where $p_{l+1}, p_{l+2}, \ldots, p_{l+N}$ are integers between 0 and $n$.

As there are an infinite number of nonzero $k_i$, then there are an infinite number of $p_i$, $p_i$ being strictly smaller than $n$.

Let $N \rightarrow \infty$, $\mathscr{H}^s(F \cap (F + t)) = 0$.

We can resume the above result as one conclusion of the following theorem.

**Theorem 2.** If $F$ is an $(m, n)$ comb Cantor set, it has a Hausdorff measure discrete spectrum given by:

$$\left(\frac{1}{n}\right)^{i_1} \ldots \left(\frac{2}{n}\right)^{i_k} \ldots \left(\frac{n-1}{n}\right)^{i_{n-1}}$$

where $i_1, i_2, \ldots, i_{n-1} \in \mathbb{Z}^+$. And if we note

$$T_{i_1, \ldots, i_{n-1}} = \left\{t : \mathscr{H}^s(F \cap (F + t)) = \left(\frac{1}{n}\right)^{i_1} \ldots \left(\frac{n-1}{n}\right)^{i_{n-1}}, \ i_1 + i_2 + \ldots + i_{n-1} = l\right\}.$$ 

We can give the exact expression of $T_{i_1, \ldots, i_{n-1}}$ as

$$T_{i_1, \ldots, i_{n-1}} = \bigcup_{i_1, \ldots, i_{n-1}} \left\{(-1)^{i_1} \frac{2d_1}{m^1} + (-1)^{i_2} \frac{2d_2}{m^{1+2}} + \ldots + (-1)^{i_{n-1}} \frac{2d_{n-1}}{m^{1+2+\ldots+n}}\right\}$$

where $A = \{1\}^{l_{n-1}} \times \{2\}^{l_{n-2}} \times \{n-1\}^{i_1}$, $\sigma(A)$ is the permutation group on $A$. 


Theorem 3. If \( F \) is the attractor of an IFS composed by a family of similarities \( \{ S_i \}_{i=1}^n \), where

\[
S_i : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \frac{1}{m}x + c_i
\]

where \( c_{i+1} - c_i = c > 0 \) for \( i = 1, \ldots, n-1 \), \( c \) is a constant and \( m \in \mathbb{Z}^+ \), \( m \geq 2 \). Particularly, when \( c_1 = 0,c_n = 1 - (1/m) \), it is one of the so called uniform Cantor set—usually built on the unit interval—when \( n/m < 1 \), and simply a segment when \( n/m \geq 1 \). The next theorem gives a rule to transform each Cantor set from the family of sets defined above to an equivalent comb Cantor set. This equivalence is taken in the sense that they are geometrically similar.

Theorem 3. If \( F \) is the attractor of an IFS composed by a family of similarities \( \{ S_i \}_{i=1}^n \), where

\[
S_i : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto ax + c_i
\]

\([a] < 1\) and \( c_{i+1} - c_i = constant > 0 \) for \( i = 1, \ldots, n-1 \). Then the geometric structure of \( F \) is independent of the values of \( c_i \), up to a scaling.

The above result for the comb Cantor sets will help us to estimate the discrete spectrum for a larger family of Cantor sets defined through the following IFS

\[C_n = \bigcup_{i \in \mathbb{Z}^+} \left\{ \frac{i}{n} \right\}
\]

where \( i/n \) is the comb Cantor set having a similar structure (in the classical mathematical meaning of similarity). If we normalize the Hausdorff measure of the set at its simplest, are illustrated for the third Cantor sets. Since they are solely based on the similarity properties, these methods can be easily applied for any fractal set when similarities are verified.

4. Numerical construction of the Hausdorff measure spectrum function

In the last section, we determined theoretically the HMSF for uniform Cantor sets. However, a direct computation of HMSF is time consuming and laborious. A natural question arises if there exists some easy ways to construct the HMSF of a fractal set. For this purpose, we discuss here three approximation algorithms which, in a concern of simplicity, are illustrated for the third Cantor sets. Since they are solely based on the similarity properties, these methods can be easily applied for any fractal set when similarities are verified.

4.1. Similarity algorithm

This algorithm is based upon the similarity properties of a fractal set which are inherited by the HMSF itself. This similarity can be obtained by a simple check of the HMSF.
Let $C$ be the triadic Cantor set generated from the unit interval $[0, 1]$, and consider the intersection of $C$ with its translation $C + t$, that we denote by $I(t) = C \cap (C + t)$, where $t$ varies between $-1$ and $1$.

Let $C = C_1 \cup C_2$, where $C_1$, $C_2$ are respectively the identical left and right parts of $C$. Clearly, $C_2 = C_1 + 2/3$ and we have

$$C \cap (C + t) = (C_1 \cup C_2) \cap ((C_1 + t) \cup (C_2 + t)) = [C_1 \cap (C_1 + t)] \cup [C_1 \cap (C_2 + t)] \cup [C_2 \cap (C_1 + t)] \cup [C_2 \cap (C_2 + t)].$$

If we use $\mathcal{M}(t)$ to express the Hausdorff measure of the intersection $I(t)$ at Hausdorff dimension $s = \log 2/\log 3$, for the right four terms, we have

$$\mathcal{M}_1 = \mathcal{H}^s(C_1 \cap (C_1 + t)) = \frac{1}{2} I(3t)$$

$$\mathcal{M}_2 = \mathcal{H}^s(C_1 \cap (C_2 + t)) = \frac{1}{2} I(3\left(t + \frac{2}{3}\right))$$

$$\mathcal{M}_3 = \mathcal{H}^s(C_2 \cap (C_1 + t)) = \frac{1}{2} I(3\left(t - \frac{2}{3}\right))$$

$$\mathcal{M}_4 = \mathcal{H}^s(C_2 \cap (C_2 + t)) = \frac{1}{2} I(3t).$$

Finally, we obtain the similarity equation verified by the HMSF of the triadic Cantor set $C$.

$$\mathcal{M}(t) = \frac{1}{2} \mathcal{M}(3\left(t - \frac{2}{3}\right)) + \mathcal{M}(3t) + \frac{1}{2} \mathcal{M}(3\left(t + \frac{2}{3}\right)).$$

If we denote by $G$ the graph of $\mathcal{M}(t)$, it is easy to see that $G$ is the attractor of three affine maps: $S_0$, $S_1$, $S_2$, i.e.

$$G = S_0(G) \cup S_1(G) \cup S_2(G)$$

where

$$S_0(x, y) = \left(\frac{1}{3}x, 0\right) \left(x, y\right)$$

$$S_1(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y\right) - \left(\frac{2}{3}, \frac{2}{3}\right) + \left(\frac{1}{3}, 0\right)$$

$$S_2(x, y) = \left(\frac{1}{3}x, \frac{1}{3}y\right) + \left(\frac{1}{3}, 0\right).$$

The process of the similarity algorithm is shown in Fig. 1.

### 4.2. Interpolation algorithm

As we see, HMSF is symmetric and discontinuous everywhere [11], but it can be approximated by a sequence of continuous functions.

We know that the triadic Cantor set is given by $C = \bigcap_{n=0}^{\infty} C_n$, and $C_{n+1}$ is obtained by deleting the middle thirds of subintervals of $C_n$.

To construct the sequence of the approximating functions of $\mathcal{M}(t)$, we always let the initial unit weight be conserved through the generalization and uniformly distributed on each subinterval in each generation. We denote these measures as $m_n$ for the $n$th generation. Then, we construct a sequence of measure functions of $C_n \cap (C_n + t)$ and we have

$$\mathcal{M}_n(t) = m_n(C_n \cap (C_n + t)).$$

We have $\mathcal{M}_n(t) \rightarrow \mathcal{M}(t)$.

Indeed, if we denote by $T_n$ and $T$ the graph of $\mathcal{M}_n(t)$ and $\mathcal{M}(t)$ respectively, denote the closure of $T$ by $T'$, which is the attractor of the three affine maps whose union will be $S = \bigcup_{n=0}^{\infty} S_n$. Using the Banach contraction theorem, we can assert that $T_{n+1} = S(T_n) \rightarrow T$ in the sense of the Hausdorff metric when $n$ goes to infinity.

The approximation is shown in Fig. 2.
Fig. 1. The generation of the HMSF by way of similarity in the generations: 0, 1, 3, 6.

Fig. 2. The approximation of the HMSF by a sequence of continuous functions, the generations: 0, 1, 3, 6.
This approximation by continuous function allows for an interpolation based method to construct the HMSF and the HMSF of a real fractal should be based on this method. Also several step checking is enough to determine the interpolation. The detail of the algorithm for the triadic Cantor set is as follows.

At the initial state, we have the abscissas $x_0^{(0)} = -1$, $x_1^{(0)} = 0$, $x_2^{(0)} = 1$, we have $M(x_0^{(0)}) = M(x_2^{(0)}) = 0$, $M(x_1^{(0)}) = 1$. In the first step, we let

$$x_0^{(1)} = x_0^{(0)}(-1);$$
$$x_1^{(1)} = \frac{2}{3} x_0^{(0)} + \frac{1}{3} x_1^{(0)} = -\frac{2}{3};$$
$$x_2^{(1)} = \frac{1}{3} x_0^{(0)} + \frac{2}{3} x_1^{(0)} = -\frac{1}{3};$$
$$x_3^{(1)} = x_1^{(0)}(0);$$
$$x_4^{(1)} = \frac{2}{3} x_1^{(0)} + \frac{1}{3} x_2^{(0)} = \frac{1}{3};$$
$$x_5^{(1)} = \frac{1}{3} x_1^{(0)} + \frac{2}{3} x_2^{(0)} = \frac{2}{3};$$
$$x_6^{(1)} = x_2^{(0)}(1).$$

In the same manner, if $x_0^{(n)}, x_1^{(n)}, \ldots, x_{2^{n+1}-1}^{(n)}$ are obtained, then, for any subsequent $x_i^{(n)}, x_{i+1}^{(n)}$, we do the following interpolation:

$$M\left(\frac{2}{3} x_i^{(n)} + \frac{1}{3} x_{i+1}^{(n)}\right) = \frac{1}{2} M(x_i^{(n)}), \quad M\left(\frac{1}{3} x_i^{(n)} + \frac{2}{3} x_{i+1}^{(n)}\right) = 0, \quad \text{if} \quad M(x_i^{(n)}) = 0;$$
$$M\left(\frac{2}{3} x_i^{(n)} + \frac{1}{3} x_{i+1}^{(n)}\right) = 0, \quad M\left(\frac{1}{3} x_i^{(n)} + \frac{2}{3} x_{i+1}^{(n)}\right) = \frac{1}{2} M(x_i^{(n)}), \quad \text{if} \quad M(x_i^{(n)}) \neq 0.$$

It is easy to see that the interpolation converges to the real function $M(t)$. Fig. 2 also illustrates this interpolation procedure.

### 4.3. Recursive algorithm

Sometimes, we only need to know the value of the HMSF at a point, the recursive algorithm is more favorable. We can write $M(t)$ in the following recursive forms:

$$M(t) = \begin{cases} M(\frac{|t| - 2/3}{2}) & \text{if} \quad 1/3 \leq |t| \leq 1; \\
M(\frac{|t| - 2/3^2}{2}) & \text{if} \quad 1/3^2 \leq |t| \leq 1/3; \\
\cdots \\
M(\frac{|t| - 2/3^k}{2}) & \text{if} \quad 1/3^k \leq |t| \leq 1/3^k; \\
\cdots 
\end{cases}$$

knowing $M(0) = 1$ and $-1 \leq t \leq 1$.

Then given any $t$, one can use this recursive form to quickly determine the numerical value of $M(t)$ in any precision.

### 5. Application of the classification method

Now that we know how to approximate the HMSF by different methods, let us show how to use it to differentiate between sets having the same fractal dimension ($\log 2/\log 3$ in this example). These sets are constructed from the initiator $I = [0, 1]$ and their respective generators are defined by the following IFS:

(a) $S_i : x \rightarrow \frac{1}{3} x + \frac{2(i - 1)}{3}$ for $i = 1, 2$;

(b) $S_i : x \rightarrow \frac{1}{9} x + \frac{8(i - 1)}{27}$ for $i = 1, 2, 3, 4$;
(c) $S_i : x \mapsto \frac{1}{9} x + \frac{2(i - 1)}{9}$ for $i = 1, 2$;

$S_i : x \mapsto \frac{1}{9} x + \frac{3i - 4}{9}$ for $i = 3, 4$;

(d) $S_i : x \mapsto \frac{1}{9} x + \frac{2(i - 1)}{9}$ for $i = 1, 2, 3, 4$;

(e) $S_1 : x \mapsto \frac{1}{9} x$, $S_2 : x \mapsto \frac{1}{9} x + \frac{2i - 1}{9}$ for $i = 2, 3$, and $S_4 : x \mapsto \frac{1}{9} x + \frac{8}{9}$.

Fig. 3 shows the HMSF of the above fractal sets. We propose here two different ways to exploit the HMSF in order to distinguish between these sets. With TIBM, we take the translation invariant values of HMSF corresponding to values preserved by translation. Each value corresponds to a level, which is a set of points representing a fixed HMSF value for different shifts (see Fig. 1). The graph of these levels, in terms of the shift number, are illustrated in Fig. 4 (all the measures have been normalized). We see that TIBM succeeds in distinguishing between (a), (c) and (b) (or (d)). However, TIBM levels are the same for (a) and (e) as well as for (b) and (d). This last fact does not allow one to conclude that (a) and (e) or (b) and (d) are the same. We have yet to go a further step in our exploration and use the FLBM. This method compares, for a given level, the HMSF values of the concerned sets. In fact, the first level, which contains the whole information of HMSF, is enough. In Fig. 5, we plotted the first four fixed levels (from 0 to 3) of the HMSF of (a) and (e). Graphically, the difference is already obvious on level one. This difference can be quantified by averaging weighted distances between shift values and the shift accumulation point at the first level, giving thus level indexes associated to each set. For example, using a dyadic sequence weights $\{1/2^i\}$, we get the value 0.5962 for (a) and 0.6248 for (e). This index, from one part, is able to differentiate between sets and, from the other part, indicates the degree of homogeneity of the set: the higher the index, the more homogeneous is the set. Finally, for sets (b) and (d), one observes that they have the same HMSF once the support corresponding to (d) is reported, by a rescaling centered on 0, on the same support of (b), Fig. 3. This bilateral matching is naturally also reflected in the FLMB.
In summary, this paper generalizes the results obtained on intersection of triadic Cantor sets [11] and uses them for classification. In this frame, we have dealt with Cantor sets having the same fractal dimension and exploited the discreteness of their HMSF to distinguish between them. In the future, we want to pursue this study on more general fractal sets.

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