Two simple ansätze for obtaining exact solutions of high dispersive nonlinear Schrödinger equations

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Abstract

We propose two simple ansätze that allow us to obtain different analytical solutions of the high dispersive cubic and cubic-quintic nonlinear Schrödinger equations. Among these solutions we can find solitary wave and periodic wave solutions representing the propagation of different waveforms in nonlinear media.

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1. Introduction

In many practical physics problems, the resulting nonlinear wave equations of interest are nonintegrable. In some particular cases they may be close to an integrable one. In these cases there exist several techniques available in order to solve approximately the problem e.g. perturbation and variational procedures and numerical analysis. However, it is always useful and desirable to construct exact analytical solutions (for instance, in order to check numerical methods).

In recent years, there has been a growing interest in developing and applying a large variety of analytical methods capable to deal with nonlinear partial differential equations of all kinds e.g. double sine-Gordon equation, Dodd–Bullough–Mikhailov equation, Korteweg–de Vries equation and many others. Among these methods we can cite the coupled amplitude-phase formalism [1], hyperbolic tangent method, Bäcklund/Darboux transformations, Hirota’s method, etc. (see [2] and the references there in).

In this paper, we deal with the existence of analytical solutions for two generalized versions of the nonlinear Schrödinger equation. The importance of this equation comprises many physics areas such as nonlinear optics, biophysics, plasma theory, nuclear hydrodynamics, etc. and describes several phenomena as diverse as coalescence of droplets in first order transitions, propagation of solitonic bubbles in a bosonic gas and others. All these phenomena can be better understood with the help of exact analytical solutions when they exist for some particular values of the parameters involved into the equation.

To find exact solutions of a nonlinear Schrödinger equation usually involves to solve an initial value problem. This may be a complicated task. Moreover, it is not always possible. There are many alternative approaches to the well known inverse scattering technique (IST) which can be used to obtain closed form solutions of nonlinear partial differential equations but almost all of them are so tedious and the calculations rather cumbersome [3,4]. It is then essential to look for new techniques as simple as possible in order to find exact analytical solutions of partial differential equations of physical relevance.

Below we propose two simple ansätze for the obtention of exact analytical solutions of nonlinear Schrödinger-type equations in the presence of linear fourth order dispersion and cubic and cubic-quintic nonlinearities. What it is

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remarkable about this method is that it allows one to find both solitary wave and periodic solutions using the same procedure.

2. Theory

With all the above things in mind, the high dispersive nonlinear Schrödinger equation can be written in the form

\[ q_z = -i \beta_2 q_t + i \frac{\beta_4}{24} q_{ttt} + i \gamma |q|^2 q, \]  

where \( q(z,t) \) is the complex amplitude of the waveform, \( z \) represents a distance coordinate and \( t \) is the time. \( \beta_2, \beta_4, \) and \( \gamma \) are known constants. For example, in nonlinear optics (1) could describe the propagation of femtosecond optical pulses into a fibre when the carrier frequency is carefully chosen, being \( \beta_2 \) and \( \beta_4 \) the group velocity dispersion and fourth order dispersion, respectively; and \( \gamma \) is the nonlinearity coefficient associated to the medium [4,5].

2.1. Ansätz 1

We will begin our analysis assuming a solution of the form (chosen in so particular way as to account for stationary soliton solutions, i.e. solitons with zero group velocity)

\[ q(z,t) = f(t) \exp(i\kappa z), \]  

where the function \( f \) has to be determined and \( \kappa \) is a constant. Substituting (2) into (1) and removing the exponential term we obtain

\[ \kappa f = -\frac{\beta_2}{2} f_t + \frac{\beta_4}{24} f_{ttt} + \gamma f^3. \]  

If we consider carefully the above expression, we can see that a linear combination of the second and fourth derivatives must provide a cubic polynomial in \( f \). Then a suitable ansätz for this can be

\[ f_t = a_0 + a_1 f + a_2 f^2, \]  

where \( a_0, a_1, \) and \( a_2 \) are constants to be determined as we shall see later.

It is straightforward to show that if we calculate \( f_{ttt} \) with the help of the above ansätz a cubic polynomial in \( f \) is obtained.

Integrating (4) we can write the following form for \( f \);

\[ f_t = \sqrt{2a_0 f + a_1 f^2 + \frac{2}{3}a_2 f^3 + K}, \]  

being \( K \) an arbitrary constant of integration.

Now, although many different combinations of null values of the arbitrary constants may arise, we will only consider several, for brevity.

- **Case 1**: \( a_0 = a_2 = K = 0 \)
  
  In this case the solution for \( f \) is an exponential of the form
  
  \[ f = \exp(\pm \sqrt{a_1} t). \]  

- **Case 2**: \( a_0 = a_1 = K = 0 \)
  
  A rational solution is obtained, namely
  
  \[ f = \frac{6}{a_2 t^2}. \]  

- **Case 3**: \( a_0 = a_1 = a_2 = 0 \)
  
  \[ f = \sqrt{K} t. \]

- **Case 4**: \( a_1 = a_2 = 0 \)
  
  \[ f = -\frac{1}{2} \frac{K - a_0^2 t^2}{a_0}. \]
Case 5: \( a_0 = K = 0 \)

The following solitary wave solution appears provided \( a_1 > 0 \)

\[
f = -\frac{3}{2} \frac{a_1}{a_2} \text{sech}^2 \left( \frac{\sqrt{a_1}}{2} t \right). \tag{10}\]

Case 6: \( a_1 = 0 \)

Now, we obtain the Weierstrass elliptic function \((a_2 > 0)\) of the form

\[
f = \text{Weierstrass} P \left( \sqrt{\frac{a_2}{6}} g_2, g_3 \right), \tag{11}\]

where \( g_2 = -12a_0/a_2 \) and \( g_3 = -6K/a_2 \).

There exist other different solutions such as secant functions, hyperbolic sine and cosine functions, etc. but we will omit them for simplicity.

Many of these solutions do not possess a clear physical significance but, however, others like those presented in Case 5 or in Case 6 do represent real physical situations such as the propagation of a femtosecond optical soliton or an unchanging periodic signal, respectively in a nonlinear medium characterized by an index of refraction given by

\[
n = n_0 + n_2 I, \tag{12}\]

where \( n_0 \) is the linear index of refraction, \( n_2 \) is the nonlinear index of refraction and \( I \) is the intensity of the light injected into the medium.

Next, in order to determine the unknown coefficients the above solutions must be substituted in the original nonlinear equation (1). We will do this, as example, for the solitary wave solution given in Case 5. Equating the coefficients of the same power in \( f \) it is very simple to obtain the following system of equations

\[
-\frac{\beta_2}{2} a_1 + \frac{\beta_4}{24} a_1^3 - \kappa = 0, \tag{13}\]

\[
-\frac{\beta_2}{2} a_2 + \frac{5\beta_4}{24} a_1 a_2 = 0, \tag{14}\]

\[
\frac{5\beta_4}{36} a_2^2 + \gamma = 0, \tag{15}\]

which solutions for \( a_1, a_2, \) and \( \kappa \) are

\[
a_1 = \frac{12}{5} \frac{\beta_2}{\beta_4}, \tag{16}\]

\[
a_2 = \pm \frac{6}{5} \frac{\sqrt{-5\beta_4}}{\beta_4}, \tag{17}\]

\[
\kappa = -\frac{24}{25} \frac{\beta_2^2}{\beta_4}. \tag{18}\]

This constitutes the well known solitary wave solution for the high dispersive cubic nonlinear Schrödinger equation describing the propagation of a femtosecond optical soliton in a fibre [6–8].

2.2. Ansätze 2

Let us now consider the high dispersive cubic-quintic nonlinear Schrödinger equation given by

\[
q_t = -i \frac{\beta_2}{2} q_x + i \frac{\beta_4}{24} q_{xx} + i \sqrt{1} |q|^2 q + i \sqrt{2} |q|^4 q. \tag{19}\]

This equation describes the propagation of a femtosecond pulse in a medium characterized by a parabolic index of refraction of the form

\[
n = n_0 + n_2 I + n_3 I^2, \tag{20}\]

being \( n_3 \) related to the nonlinear fourth order susceptibility.
If we proceed in a way analogous to the case of the cubic equation it appears evident that a new ansatz is needed to take account of the quintic term. So in order to get several analytical solutions let us try the following ansatz

\[ f_t = a_0 + a_1 f + a_2 f^2 + a_3 f^3. \]  

This implies that the first derivative of the function \( f \) must satisfy the following relation

\[ f_t = \sqrt{2a_0 f + a_1 f^2 + \frac{2}{3} a_2 f^3 + \frac{a_3}{2} f^4 + K}. \]  

The integration of this expression for different values of the parameters provides for very many possible combinations of null arbitrary constants. Some of them are shown below (also for brevity).

- **Case 1:** \( a_0 = a_1 = a_3 = K = 0 \)
  \[ f = \frac{6}{a_2} t^2. \]

- **Case 2:** \( a_0 = a_2 = a_3 = K = 0 \)
  \[ f = \exp(\pm \sqrt{a_1} t). \]

- **Case 3:** \( a_1 = a_2 = a_3 = K = 0 \)
  \[ f = \frac{3}{2a_2} \sech^2 \left( \frac{\sqrt{a_1}}{2} t \right). \]

- **Case 4:** \( a_0 = a_3 = K = 0 \)
  \[ f = -\frac{3a_1}{2a_2} \sech \left( \frac{\sqrt{a_1}}{2} t \right). \]

- **Case 5:** \( a_0 = a_2 = K = 0 \)
  \[ f = -\frac{2a_1}{a_3} \sech (\sqrt{a_1} t). \]

This is the well known bright optical soliton solution [9,10].

- **Case 6:** \( a_0 = a_1 = K = 0 \)
  \[ f = \frac{12a_2}{2a_2 t^2 - 9a_3}. \]

- **Case 7:** \( a_1 = a_3 = 0 \)
  \[ f = \text{Weierstrass } P \left( \sqrt{\frac{a_2}{6}}, g_2, g_3 \right), \]

with \( g_2 = -12a_0/a_2 \) and \( g_3 = -6K/a_2 \).

- **Case 8:** \( a_0 = a_2 = 0 \)

In this case there exist several possibilities depending on the values of the constant \( K \). For example, if \( K = a_1^2/2a_3 \) then the solution for \( f \) reads

\[ f = \sqrt{-\frac{a_1}{a_3}} \tanh \left( \sqrt{\frac{a_1}{2}} t \right). \]

which represents the dark optical soliton for the high dispersive cubic-quintic nonlinear Schrödinger equation [10,11], whereas if \( K = (1 - m^2)/(2m^2 - 1)^2 \) then \( f \) can be expressed as the following Jacobi elliptic function [12]

\[ f = \sqrt{-\frac{2a_1 m^2}{a_3 (2m^2 - 1)}} \operatorname{cn} \left( \sqrt{\frac{a_1}{2m^2 - 1}} t \right). \]

Finally, if \( K = a_1^2 m^2/(a_3 (m^2 + 1)) \) the solution for \( f \) is the Jacobi elliptic function

\[ f = \sqrt{-\frac{2a_1 m^2}{a_3 (m^2 + 1)}} \operatorname{sn} \left( \sqrt{-\frac{a_1}{m^2 + 1}} t \right). \]
We notice that, in contrast with the first ansätz, now it appears a Lorentzian-type solution shown in Case 6. This constitutes a bell-shaped solution and also could represent a pulse propagating undistorted (a soliton).

3. Conclusions

The method proposed permits to obtain in a straight manner several particular analytical solutions of the high dispersive cubic and cubic-quintic nonlinear Schrödinger equations. We must admit that these equations are so particular and due to the special linear dependence expressed in the second and fourth order derivatives the ansätze proposed do fit very well for our purpose. However, if other types of nonlinear partial differential equations are treated with the same technique as above we can not assure the applicability. This could constitute a potential area of research. On the other hand, there exist other known different analytical solutions that can not be obtained with the help of the proposed ansätze. For example, for the case of the cubic-quintic nonlinear Schrödinger equation there are two hump soliton solutions already published [6]. It is left to future papers the search for suitable models.

References