Some simple global synchronization criterions for coupled time-varying chaotic systems

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Abstract

Based on the Lyapunov stabilization theory and matrix measure, this paper proposes some simple generic criterions of global chaos synchronization between two coupled time-varying chaotic systems from a unidirectional linear error feedback coupling approach. These simple criterions are applicable to some typical chaotic systems with different types of nonlinearity, such as the original Chua’s circuit and the Rössler chaotic system. The coupling parameters are determined according to the new criterion so as to ensure the coupled systems’ global chaos synchronization.

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1. Introduction

Recently, chaos synchronization has been widely investigated and many effective methods have been presented [1–12]. Due to the simple configuration and easy implementation, the unidirectional linear error feedback coupling scheme can be adopted in many real systems, this is one of the most efficient methods for chaos synchronization [13–19]. In order to design a response or slave chaotic system by using the unidirectional linear error feedback methodology, the choice of the feedback gain (or coupling parameters) is the key problem to be considered.

In this paper, based on Lyapunov stability theory and matrix measure, we study the synchronization of two coupled time-varying chaotic systems using the unidirectional linear error feedback scheme. The aim of this paper is to further develop some simple but generic criterions for the global synchronization of two coupled general time-varying chaotic systems, along with a simple configuration for the corresponding implementation. Some simple generic conditions of global chaos synchronization of two coupled time-varying chaotic systems are derived, and to apply the conditions to some typical chaotic systems, for example, the original Chua’s circuit and Rössler chaotic system such that synchronization is achieved.

The rest of the paper is organized as follows. In Section 2, based on Lyapunov stability theory and matrix measure, some generic conditions of global chaos synchronization concerns two coupled time-varying systems using the unidirectional linear error feedback coupling scheme, and some global chaos synchronization criterions are established. Such conditions are applied to some typical chaotic systems with different type of nonlinear functions in Section 3, such as the classic Chua’s circuit and Rössler chaotic system. Finally, conclusion remarks are then given in Section 4.

2. Some criterions for global chaos synchronization

Considering a time-varying chaotic system with state equation in the form

$$\dot{x} = A(t)x + g(x) + u$$  \hspace{1cm} (1)

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where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^n \) is the external input vector, \( A \in \mathbb{R}^n \) is a time-varying matrix and \( g(x) \) is a continuous nonlinear function.

Assuming that
\[
g(x) - g(\bar{x}) = M(x, \bar{x})(x - \bar{x})
\]
for a bounded matrix \( M(x, \bar{x}) \), in which the elements are dependent on \( x \) and \( \bar{x} \). Most of chaotic system, including all Lur’e nonlinear systems and Lipschitz nonlinear systems, can be described by (1) and (2) when \( A(t) \equiv A \).

Form the unidirectional linear coupling approach, a salve system for (1) is constructed as follows:
\[
\dot{x} = A(t)\bar{x} + g(\bar{x}) + u + K(x - \bar{x})
\]
where \( K = \text{diag}(k_1, k_2, \ldots, k_n) \), with \( k_i \in \mathbb{R}, i = 1, 2, \ldots, n \), is a feedback matrix to be designed later.

From (1) and (2), the following error system equation can be obtained:
\[
\dot{e} = A(t)e + g(x) - g(\bar{x}) - K(x - \bar{x}) = A(t)e - Ke + g(x) - g(\bar{x}) = (A(t) + M(x, \bar{x}) - K)e
\]
where \( e = x - \bar{x} \) is the error term.

**Theorem 1.** If there exists the feedback gain matrix \( K \) such that
\[
\int_{t_0}^{t} \mu(A(t) + M(x, \bar{x}) - K)dt = -\infty \quad \forall t_0 \geq 0
\]
uniformly for all \( x, \bar{x} \) in the phase space, then the error dynamical system (4) is globally asymptotically stable about origin, implying that the two systems (1) and (3) are globally asymptotically synchronized.

**Proof.** Let \( \| * \| \) be any vector norm and \( \| * \| \) be the matrix norm induced by this vector norm, the symbol \( \mu(\cdot) \) denotes the matrix measure derived from the matrix norm \( \| * \| \), let \( e(t) \) be a solution of system (4), for \( t \geq 0 \)
\[
\frac{d|e|}{dt} - \mu(A(t) + M(x, \bar{x}) - K)|e| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} ||e(t + \varepsilon) - I + \varepsilon(A(t) + M(x, \bar{x}) - K)e(t)|| \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} ||e(t + \varepsilon) - I + \varepsilon(A(t) + M(x, \bar{x}) - K)e(t)||
\]
Since Eq. (4) holds, then \( \frac{d|e|}{dt} \leq \mu(A(t) + M(x, \bar{x}) - K)|e| \) hence \( |e| \leq \exp \int_{t_0}^{t} \mu(A(s) + M(x, \bar{x}) - K) ds \) we can obtain that the condition of Theorem 1 ensures asymptotic stability of solution about system (4).

**Remark 1.** If \( \mu(A(t) + M(x, \bar{x}) - K) \leq \lambda < 0 \), then condition of Theorem 1 is satisfied.

**Corollary 1.** The two systems (1) and (3) are globally asymptotically synchronized, if there exists the feedback gain matrix \( K \) such that one condition is satisfied at least as follows:

1. \( \int_{t_0}^{\infty} \max \left\{ a_{ij}(t) + M_{ij}(x, \bar{x}) - k_j + \sum_{i \neq j} |a_{ij}(t) + M_{ij}(x, \bar{x})| \right\} dt = -\infty; \)
2. \( \int_{t_0}^{\infty} \max \left\{ |A(t) + M(x, \bar{x}) - K|^T + |A(t) + M(x, \bar{x}) - K| \right\} dt = -\infty; \)
3. \( \int_{t_0}^{\infty} \max \left\{ a_{ii}(t) + M_{ii}(x, \bar{x}) - k_i + \sum_{j \neq i} |a_{ij}(t) + M_{ij}(x, \bar{x})| \right\} dt = -\infty; \)
4. there exists \( \omega_i > 0, (i = 1, 2, \ldots, n) \) such that \( \int_{t_0}^{\infty} \max \left\{ |a_{ij}(t) + M_{ij}(x, \bar{x}) - k_i + \sum_{j \neq i} |a_{ij}(t) + M_{ij}(x, \bar{x})| \right\} dt = -\infty. \)

**Example.** Consider system
\[
\dot{x} = \begin{pmatrix} -20.4 & 25 \\ 16 & -20.4 \end{pmatrix} x
\]
Obviously, we can obtain system (5) is asymptotically stable from Theorem 1, but it is difficult that to obtain this conclusion by using results of paper [18,19].

**Theorem 2.** If there exists a positive definite symmetric constant matrix \( P \) and the feedback gain matrix \( K \) is chosen such that
\[
\int_{t_0}^{+\infty} \mu \left\{ [A(t) - K + M(x,\bar{x})]^T P + P[A(t) - K + M(x,\bar{x})] \right\} dt = -\infty
\]

then the error dynamical system (4) is globally asymptotically stable about the origin, implying that the two systems (1) and (3) are globally asymptotically synchronized.

**Proof.** Choose Lyapunov function \( V = e^T Pe \), where \( P \) is a positive definite symmetric constant matrix. Then, its derivative is:

\[
\dot{V} = \dot{e}^T Pe + e^T \dot{P} e = [A(t) - K + M(x,\bar{x})]e^T Pe + e^T P[A(t) - K + M(x,\bar{x})]e
\]

\[
= e^T \left\{ [A(t) - K + M(x,\bar{x})]^T P + P[A(t) - K + M(x,\bar{x})] \right\} e
\]

\[
\leq \mu \left\{ [A(t) - K + M(x,\bar{x})]^T P + P[A(t) - K + M(x,\bar{x})] \right\} e^T e
\]

\[
\leq \frac{\mu \left\{ [A(s) - K + M(x,\bar{x})]^T P + P[A(s) - K + M(x,\bar{x})] \right\}}{\lambda_{\text{min}}(P)} V
\]

hence

\[
e^T e \leq \frac{V(t)}{\lambda_{\text{min}}(P)} \leq \frac{V(t_0)}{\lambda_{\text{min}}(P)} \exp \int_{t_0}^{t} \frac{\mu \left\{ [A(s) - K + M(x,\bar{x})]^T P + P[A(s) - K + M(x,\bar{x})] \right\}}{\lambda_{\text{min}}(P)} ds
\]

It is known that, system (4) is globally exponentially stable, and hence, the two systems (1) and (3) are globally asymptotically synchronized. \( \square \)

**Remark 2.** In Theorem 2, if \( A(t) \) is a constant matrix and we choose special matrix measure \( \mu (\ast) \), we can obtain the results of paper [18,19]. In addition, the calculation of matrix measure is more convenient than calculation of eigenvalue of matrix.

3. Synchronization of some typical chaotic systems

To demonstrate the use of chaos synchronization criterion proposed herein, two examples of chaotic systems are considered.

3.1. The original Chua’s circuit

Chua’s circuit [20] can be described by

\[
\begin{aligned}
\dot{x} &= a(y - x - f(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y
\end{aligned}
\]

where \( a > 0, \beta > 0, a < b < 0, f(x) \) is a piecewise linear function described by

\[
f(x) = bx + \frac{1}{2} (a - b)(|x + 1| - |x - 1|)
\]

In Eq. (7), we have

\[
f(x) - f(\bar{x}) = k(x,\bar{x})(x - \bar{x})
\]

where \( k(x,\bar{x}) \) is dependant on \( x \) and \( \bar{x} \), and varies in the interval \( [a, b] \) for \( t \geq 0 \), that is, \( k(x,\bar{x}) \) is bounded by constants as \( a \leq k(x,\bar{x}) \leq b < 0 \).

Referring to Eq. (3), the following slave system is constructed for equations with linear unidirectional coupling:

\[
\begin{aligned}
\dot{\hat{x}} &= a(y - \hat{x} - f(\hat{x})) + k_1(x - \hat{x}), \\
\dot{\hat{y}} &= \hat{x} - \hat{y} + \hat{z} + k_2(y - \hat{y}), \\
\dot{\hat{z}} &= -\beta \hat{y} + k_3(z - \hat{z})
\end{aligned}
\]
Subtracting Eq. (9) from Eq. (6), we obtain
\[
\begin{align*}
\dot{e}_x &= a(e_y - e_x - k \dot{x}, \dot{t})e_x - k_1 e_x, \\
\dot{e}_y &= e_y - e_x + e_z - k_2 e_x, \\
\dot{e}_z &= -\beta e_z - k_3 e_z
\end{align*}
\] (10)
where \(e_x = x - \ddot{x}, \quad e_y = y - \ddot{y}, \quad e_z = z - \ddot{z}, \quad e = (e_x, e_y, e_z)^T.\)

Eq. (10) can be rewritten as:
\[
\dot{e} = Ae + g(x) - g(\ddot{x}) - Ke
\] (11)
where
\[
A = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -\beta \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} -\alpha f(x) \\ 0 \\ 0 \end{pmatrix}
\]

Observe that
\[
g(x) - g(\ddot{x}) = \begin{pmatrix} -\alpha f(x) - f(\ddot{x}) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha k(\ddot{x}, \ddot{x})(x - \ddot{x}) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\alpha k(\ddot{x}, \ddot{x}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = M(x, \ddot{x})e
\] (12)

where
\[
M(x, \ddot{x}) = \begin{pmatrix} -\alpha k(\ddot{x}, \ddot{x}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

From Eqs. (11) and (12), we get
\[
A - K + M(x, \ddot{x}) = \begin{pmatrix} -\alpha - k_1 - \alpha k(\ddot{x}, \ddot{x}) & \alpha & 0 \\ 1 & -1 - k_2 & 1 \\ 0 & -\beta & -k_3 \end{pmatrix}
\] (13)

One can choose
\[
\max\{1 - \alpha - \alpha k(\ddot{x}, \ddot{x}) - k_1, \alpha + \beta - 1 - k_2, 1 - k_3\} < 0 \quad \text{or} \quad \{-\alpha k(\ddot{x}, \ddot{x}), 1 - k_2, \beta - k_3\} < 0
\] (14)

According to Theorem 1 and Corollary 1, the two coupled Chua's system (6) and (9) are globally asymptotically synchronized.

Since \(\alpha > 0\) and \(a \leq k(\ddot{x}, \ddot{x}) \leq b < 0\), from (14), one can choose
\[
\max\{1 - \alpha - \alpha a - k_1, \alpha + \beta - 1 - k_2, 1 - k_3\} < 0 \quad \text{or} \quad \{-\alpha a - k_1, 1 - k_2, \beta - k_3\} < 0
\] (15)

In addition,
\[
[A - K + M(x, \ddot{x})]^T + [A - K + M(x, \ddot{x})] = \begin{pmatrix} -2\alpha - 2\alpha k(\ddot{x}, \ddot{x}) - 2k_1 & \alpha + 1 & 0 \\ \alpha + 1 & -2 - 2k_2 & 1 - \beta \\ 0 & 1 - \beta & -2k_3 \end{pmatrix}
\]

One may then choose
\[
\max\left\{\frac{1}{2} - \frac{1}{2} - \alpha k(\ddot{x}, \ddot{x}) - k_1, \frac{1}{2} - \frac{1}{2} + \frac{1}{2} |1 - \beta| - k_2, \frac{1}{2} |1 - \beta| - k_3\right\} < 0
\] (16)

According to Theorem 2, the two coupled Chua's system (6) and (9) are globally asymptotically synchronized. From (16), one can choose
\[
\max\left\{\frac{1}{2} - \frac{1}{2} - \alpha a - k_1, \frac{1}{2} - \frac{1}{2} + \frac{1}{2} |1 - \beta| - k_2, \frac{1}{2} |1 - \beta| - k_3\right\} < 0
\] (17)
So that slave system of \((18)\) is constructed as follows:

\[
\begin{align*}
\dot{x} &= -(y + z), \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x - c)
\end{align*}
\] (18)

where \(a, b\) and \(c\) be positive parameter. According to the unidirectional linear error feedback coupling approach, the slave system of (18) is constructed as follows:

\[
\begin{align*}
\dot{\tilde{x}} &= -(\tilde{y} + \tilde{z}) + k_1(x - \tilde{x}), \\
\dot{\tilde{y}} &= \tilde{x} + a\tilde{y} + k_2(y - \tilde{y}), \\
\dot{\tilde{z}} &= b + \tilde{z}(x - c) + k_3(z - \tilde{z})
\end{align*}
\] (19)

It follows from (18) and (19) that

\[
\dot{e} = Ad + g(x) - g(\tilde{x}) - Ke
\]

where

\[
A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad e = \begin{pmatrix} x - \tilde{x} \\ y - \tilde{y} \\ z - \tilde{z} \end{pmatrix} \quad \text{and} \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ xz \end{pmatrix}.
\]

Hence, one has \(g(x) - g(\tilde{x}) = M(x, \tilde{x})e\) and

\[
M(x, \tilde{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{z} & 0 & x \end{pmatrix}
\]

so that

\[
A - K + M(x, \tilde{x}) = \begin{pmatrix} -k_1 & -1 & -1 \\ 1 & a - k_2 & 0 \\ \tilde{z} & 0 & x - c - k_3 \end{pmatrix}
\]

It follows from Theorems 1 and 2 that if

\[
\max\{2 - k_1, 1 + a - k_2, x - c - |\tilde{z}| - k_3\} < 0
\] (20)

or

\[
\max\left\{\frac{1}{2}|\tilde{z} - 1| - k_1, a - k_2, x - c + \frac{1}{2}|\tilde{z} - 1| - k_3\right\} < 0
\] (21)

then the two coupled Rössler systems (18) and (19) are globally asymptotically synchronized.

**Corollary 3.** For the two coupled Rössler systems (18) and (19), if \(k_1, k_2, k_3\) are chosen such that the inequality (20) or (21) holds, then they are globally asymptotically synchronized.
Remark 3. Since the trajectory of a chaotic system is bounded, inequality (20) or (21) holds for large enough values of $k_1, k_2, k_3$.

4. Conclusions

In this paper, using matrix measure, some simple criterions are derived for the global synchronization of two coupled general chaotic systems with a unidirectional linear error feedback coupling. Suitable coupling parameters can be easily designed accounting to the given condition to ensure the global chaos synchronization. These simple criterions are also demonstrated that they can be applied to chaotic systems with different types of nonlinearity.

References