Stability analysis of cellular neural networks with nonlinear dynamics

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1. Introduction

Cellular neural networks (CNNs), introduced by Chua and Yang [1], offer a new approach to real-time signal processing.

Definition 1. The CNN is a
(i) 2-, 3-, or \( n \)-dimensional array of
(ii) mainly identical dynamical systems, called cells, which satisfies two properties:
(iii) most interactions are local within a finite radius \( r \), and
(iv) all state variables are continuous valued signals.

In this paper we show that by retaining all important features of the original CNN structure [1] and by introducing nonlinearity in the templates, the CNN becomes a powerful framework for general analogue array dynamics.
Let us consider the following dynamical system for CNN. As a state equation we will take one with nonlinear cloning templates:

\[
C \dot{v}_{ij} = -\frac{1}{R_x} v_{ij}(t) + \sum_{C(k, l) \in N(i, j)} A(v_{kl}(t), v_{ij}(t)) + \sum_{C(k, l) \in N(i, j)} B(v_{kl}, v_{ij}), \quad 1 \leq i \leq M, \quad 1 \leq j \leq N,
\]  

(1.1)

\(v_{ij}, v_{ij}, v_{ij}\) refer to the state, output and input voltage of a cell \(C(i, j)\); \(C, R_x\) are fixed values of a linear capacitor and a linear resistor, we will assume that \(C > 0, R_x > 0; A(v_{ij}, v_{ij})\) defines nonlinear feedback cloning template, \(B(v_{ij}, v_{ij})\) is a nonlinear control cloning template, called as well – nonlinear feedback and control operator, respectively. A template specifies the interaction between each cell \(C(i, j)\) and its neighbor cells \(C(k, l)\) in terms of their input, state, and output variables. The structure of the nonlinearity in the templates is also important: it is a function of at most two variables, namely the output voltage of the cell \(C_{ij}\) and that of a neighbor \(C_{kl}\). We will take \(A\) and \(B\) in the following form:

\[
A = \begin{bmatrix}
0 & p_1 & 0 \\
p_2 & 2 & p_2 \\
0 & p_1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 \\
p_3 & 1 & p_3 \\
0 & 0 & 0
\end{bmatrix}
\]

with \(p_1 = c_1 v_{kl} v_{ij}; p_2 = c_2 \exp(v_{kl}); p_3 = c_3 (v_{kl} + v_{ij}).\)

We will allow the output function to have its own dynamics, instead of piecewise-linear function considered in [1]. Moreover, we will consider output equation of integro-differential type:

\[
\dot{v}_{ij} = -v_{ij} + \int_0^1 f(v_{ij}(t)) \, dt,
\]  

(1.2)

where \(f(x)\) might be any smooth \((C^1)\) strictly monotone-increasing sigmoid function [1]. This is sometimes desirable in analytical proofs where \(C^1\) condition is more convenient to work with. It is also a more realistic assumption since the physically realized characteristic is actually \(C^1\). Let us consider \(f\) in the following form:

\[
f(v_{ij}(t)) = \begin{cases}
0, & v_{ij} < 0, \\
v_{ij}(t), & 0 \leq v_{ij} \leq 1, \\
1, & v_{ij} > 1.
\end{cases}
\]

The input equation is

\[
v_{ij} = E_{ij}, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N,
\]  

(1.3)

where \(E_{ij}\) is an input source.

The initial and the boundary conditions are the following:

\[|v_{ij}(0)| \leq 1,\]  

(1.4)
\[ |v_{uij}| \leq 1, \quad (1.5) \]

\[ |v_{uij}(0)| \leq 1. \quad (1.6) \]

The proposed architecture of a CNN is quite general. The nonlinear cloning templates allow us to model some biological problems, and also it can be used for modeling motion dynamics. The output equation (1.2) gives more possibilities for complex dynamics and interesting phenomena. It is very useful for the circuit layout and the software programmability of a CNN described above.

In Section 2 of this paper we will define dynamic range of a CNN and we will make steady-state analysis. Section 3 deals with perturbed CNN [5]. The solutions for such CNN are constructed by applying the method of Lyapunov’s majorizing equations. In Section 4 we will give some numerical results for this quite general CNN architecture.

2. Dynamic properties and stability

2.1. Dynamic range

First of all, we will give an estimate for the dynamic range of the CNN defined above.

**Proposition 1.** For a CNN described by the bounded nonlinear cloning templates (1.1) and the special case of output (1.2), all states \( v_{uij} \) are bounded for all \( t > 0 \) and the bound \( v_{\text{max}} \) can be computed by the following formula:

\[
 v_{\text{max}} = 1 + R_x \max_{1 \leq i \leq M, 1 \leq j \leq N} \left[ \sum_{C(k,l) \in N_i(i,j)} (\max_{t} |A| + \max_{u} |B|) \right].
\]  

(2.1)

**Proof.** Let us consider the state equation of a cell (1.1) in the form

\[
 \dot{v}_{uij} = -\frac{1}{R_x C} v_{uij}(t) + F_{ijkl}(t) + G_{ijkl}(u), \quad 1 \leq i \leq M, 1 \leq j \leq N,
\]

(2.2)

where

\[
 F_{ijkl}(t) = \frac{1}{C} \sum_{C(k,l) \in N_i(i,j)} A(v_{ijkl}(t), v_{ijkl}(t)),
\]

\[
 G_{ijkl}(u) = \frac{1}{C} \sum_{C(k,l) \in N_i(i,j)} B(v_{ijkl}(t), v_{uij}(t)).
\]

Eq. (2.2) is a first-order ordinary differential equation and its solution is [2]

\[
 v_{uij}(t) = v_{uij}(0) e^{-t/R_x C} + \int_{0}^{t} e^{-(t-\tau)/R_x C} [F_{ijkl}(\tau) + G_{ijkl}(u)] d\tau.
\]  

(2.3)
Therefore,
\[ |v_{xij}(t)| \leq |v_{xij}(0)|e^{-t/R,C} + \int_0^t e^{-(t-\tau)/R,C} [|F_{ijkl}| + |G_{ijkl}(u)|] \, d\tau \]
\[ \leq |v_{xij}(0)|e^{-t/R,C} + [F + G] \int_0^t e^{-(t-\tau)/R,C} \, d\tau \]
\[ \leq |v_{xij}(0)| + R,C[F + G], \]
where
\[ F \equiv \max_t |F_{ijkl}(t)| \leq \frac{1}{C} \sum_{C(k,l) \in \mathcal{N}_i} \max_t |A(v_{ijkl},v_{xij})|, \]
\[ G \equiv \max_u |G_{ijkl}| \leq \frac{1}{C} \sum_{C(k,l) \in \mathcal{N}_i} \max_u |B(v_{ijkl},v_{xij})|. \]

From conditions (1.4)–(1.6), it follows that
\[ |v_{xij}(t)| \leq 1 + R_x \left[ \sum_{C(k,l) \in \mathcal{N}_i} \left( \max_t |A| + \max_u |B| \right) \right], \]
\[ 1 \leq i,k \leq M, \quad 1 \leq j,l \leq N. \quad (2.4) \]

Now let
\[ v_{\text{max}} = \max_{(i,j)} \left\{ 1 + R_x \sum_{C(k,l) \in \mathcal{N}_i} \left( \max_t |A| + \max_u |B| \right) \right\}. \quad (2.5) \]

Then, we have
\[ \max_t |v_{xij}| \leq v_{\text{max}} \]
for all \( 1 \leq i,k \leq M, \quad 1 \leq j,l \leq N. \)

For any cellular neural network parameters \( R_x, C \) are finite constants, estimated by formula (2.1).

**Remark 1.** The main difference between CNNs and neural networks (NNs) is that a CNN has practical dynamic range, which can be calculated by formula (2.1). Whereas, general NN often suffers from a severe dynamic range restrictions in the circuit implementation stage.

### 2.2. Stability of a CNN with nonlinear cloning templates

CNN defined mathematically by dynamical system (1.1)–(1.6) can operate in two modes: transient mode or dc steady-state mode. In transient mode the output is the snapshot at a given time \( t = T \). In the dc steady-state mode the output is the dc steady state of the circuit.
Theorem 1. For a CNN, described by the dynamical system (1.1)–(1.6), we always obtain a constant output after the transient has decayed to zero. In other words, we have
\[
\lim_{t \to \infty} v_{\text{ij}}(t) = \text{const.}, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N.
\] (2.6)

Proof. According to the characteristic of a function \(f(v_{\text{ij}})\) there are four possible cases for the state of a cell \(C(i,j)\):

1. \(v_{\text{ij}}(t) < 0\),
2. \(0 \leq v_{\text{ij}}(t) \leq 1\), \(dv_{\text{ij}}/dt = 0\),
3. \(0 \leq v_{\text{ij}}(t) \leq 1\), \(dv_{\text{ij}}/dt \neq 0\),
4. \(v_{\text{ij}}(t) > 1\).

Output equation (1.2) is a first-order integro-differential equation and its solution is [7,8]
\[
v_{\text{ij}}(t) = v_{\text{ij}}(0)e^{-t} + \int_0^t e^{-(t-\zeta)} \int_0^1 f(v_{\text{ij}}(\zeta)) d\zeta.
\] (2.7)
If we change the order of integration in the second addend we get
\[
v_{\text{ij}}(t) = v_{\text{ij}}(0)e^{-t} + \int_0^1 f_1(v_{\text{ij}}(t)) dt,
\] (2.8)
where
\[
f_1(v_{\text{ij}}(t)) = \int_0^t e^{-(t-\zeta)} f(v_{\text{ij}}(\zeta)) d\zeta.
\]
Now, let us consider the first case, when \(v_{\text{ij}}(t) < 0\). Then, \(f(v_{\text{ij}}) = 0 \Rightarrow \lim_{t \to \infty} v_{\text{ij}}(t) = 0\).

In the fourth case, we have \(v_{\text{ij}}(t) > 1\) and \(f(v_{\text{ij}}(t)) = 1\), therefore
\[
\lim_{t \to \infty} v_{\text{ij}}(t) = 1.
\]
If we consider the second case, \(0 \leq v_{\text{ij}}(t) \leq 1\), then \(f(v_{\text{ij}}) = v_{\text{ij}} = \text{const.} = C \in [0, 1]\). We obtain form (2.8):
\[
\lim_{t \to \infty} v_{\text{ij}}(t) = C = \text{const.}
\]
In the third case \(v_{\text{ij}} \neq \text{const.}\), so we can easily prove that
\[
\lim_{t \to \infty} v_{\text{ij}}(t) \to v_{\text{max}},
\]
which is still a constant.

Theorem 1 is proved. □

The main property of a CNN [1] is that every cell has the same connections as its neighbors. Therefore, each cell’s circuit equation is the same as that of the other cells in the same circuit. Hence, we can understand the global properties of a CNN by studying the local properties of a single cell. For this reason we will first
define stable cell equilibrium points. Let us denote: $v_{xij}(t) \equiv x$, $v_{yij}(t) \equiv y$, $v_{uij} \equiv u$,

\[ \sum_{C(k,l) \in N(i,j)} A(v_{xkl}, v_{yij}) \equiv A(y) \), \sum_{C(k,l) \in N(i,j)} B(v_{xkl}, v_{uij}) \equiv B(u) \). Then, the following dynamical system is considered:

\[ \dot{x} = -\frac{1}{R_x} x + A(y) + B(u) \equiv F_1(x, y, u), \]
\[ \dot{y} = -y + \int_0^1 f(x(t)) \, dt \equiv F_2(x, y). \quad (2.9) \]

**Definition 2.** Equilibrium points for system (2.9) are those for which $F_1(x, y) + G(u, t) = 0$ and $F_2(x, y) = 0$, i.e.

\[ y^* = \int_0^1 f(x^*(t)) \, dt, \]
\[ x^* = R_x A(y^*) + R_u B(u). \quad (2.10) \]

We assume first that $B(u) = 0$, hence

\[ x^* = R_x A \left( \int_0^1 f(x^*(t)) \, dt \right). \]

Let $R_x = 1$ and let us consider $A$ to be in the form of the following cloning template:

\[ A = \begin{bmatrix} 0 & p_1 & 0 \\ p_2 & 2 & p_2 \\ 0 & p_1 & 0 \end{bmatrix}. \]

$p_1 = c_1 v_{xkl} v_{yij}$, $p_2 = c_2 \exp(v_{xkl})$.

According to [2] the Jacobian matrix of an equilibrium point for the dynamical system (2.9) is $J = DF_1(x^*)$ and it can be computed by

\[ J_{ij} = \frac{\partial F_1}{\partial x_j} \bigg|_{x=x^*}. \]

If all of the eigenvalues of $J$ have negative real parts, the equilibrium point $x = x^*$ is asymptotically stable. On the other side, if one of the eigenvalues of $J$ has a positive real part then the equilibrium point $x^*$ is unstable. In general, if none of the eigenvalues of $J$ have zero real part, then the equilibrium point $x^*$ is hyperbolic.

From the sigmoid characteristic of $f(x^*)$, the following three cases are possible:

1. $f(x^*) = 0$ then the stable equilibrium points are $x^* = c_1$, $x^* = 2$ for $c_1 > 1$;
2. $f(x^*) = x^*$ and $x^* = \text{const.}$. the stable equilibrium point is $x^* = 2$;
3. $f(x^*) = 1$ then the stable equilibrium points are $x^* = c_1$, $x^* = c_2 e$, $x^* = 2$, for $c_1 > 1$, $c_2 > 1$.

For the case when $B(u)$ is with constant entries the results are analogous.
Therefore, the following definitions are true.

**Definition 3** (Cell equilibrium state). A cell equilibrium state \( v_{xij}^* \) of a circuit \( C(i,j) \) in a CNN with dc input \( v_{ukl} \) is any value of the state variable \( v_{xij} \) which satisfies:

(a) \( \frac{dv_{xij}}{dt} |_{v_{xij} = v_{xij}^*} = 0 \),

(b) \( v_{xij} = \{0,1\} \), for all neighbor cells \( C(k,l) \in N_r(i,j) \).

**Definition 4** (Stable cell equilibrium states). A cell equilibrium state, \( v_{xij}^* \), of a cell circuit \( C(i,j) \) is said to be stable iff 

\[ |v_{xij}^*| > 1. \]

Let us summarize the above observations.

**Theorem 2.** Each cell of the general nonlinear CNN (1.1)–(1.6) must settle at a stable equilibrium point in its dc steady state, if \( v_{xij}^* < 0 \) or \( v_{xij}^* > 1 \) and \( \frac{dv_{xij}}{dt} |_{v_{xij} = v_{xij}^*} = 0 \), \( 1 \leq i \leq M, \ 1 \leq j \leq N \).

Moreover, the magnitude of all stable cell equilibrium points is greater than 1, i.e.

\[ \lim_{t \to \infty} v_{xij}(t) > 1, \quad 1 \leq i \leq M, \ 1 \leq j \leq N \]

and

\[ \lim_{t \to \infty} v_{yij}(t) = \{0,1\}, \quad 1 \leq i \leq M, \ 1 \leq j \leq N. \]

Now we will define stable system equilibrium points for our CNN which give its global dynamic behavior.

**Definition 5** (Stable system equilibrium points). A stable system equilibrium point of a CNN described by the dynamical system (1.1)–(1.6) is a state vector with components, \( v_{xij}^* \), which are stable cell equilibrium points defined above.

**Remark 2.** Since any stable system equilibrium point is a limit point of a set of trajectories of the corresponding system of differential and integro-differential equations (2.9), such an attracting limit point is said to have a basin of attraction [2].

3. Dynamic behavior of the perturbed CNN

In the previous sections we have proved that a nonlinear CNN, defined by the dynamical system (1.1)–(1.6), must always converge to a constant steady state after the transient has decayed to zero.

We introduce now a small parameter \( \mu \), which can come from some noise sources of known statistics in the output circuit. The dynamical system in this case is of the type

\[ C\ddot{x} = -\frac{1}{R_x} x + \mu A(y) + B(u) \]
\[
\dot{x} = \frac{1}{R_x} x + U(t, y, u, \mu),
\]

\[
\dot{y} = -y + \int_0^1 f(x(t)) \, dt = -y + V(t, x), \tag{3.1}
\]

\[
U(t, y, u, \mu) = \mu A(y) + B(u), \quad V(t, x) = \int_0^1 f(x(t)) \, dt.
\]

We will construct the solutions of the system (3.1) by applying the method Lyapunov’s majorizing equations [4,6].

An auxiliary system for system (3.1) is the following:

\[
\begin{align*}
\dot{x} &= -\frac{1}{R_x} x + \varphi(u, t), \\
\dot{y} &= -y + \psi(t), \tag{3.2}
\end{align*}
\]

where \( \varphi(u, t) \) and \( \psi(t) \) are arbitrary \( C^1 \) functions. Its solution is

\[
\begin{align*}
x &= \left[ e^{(1/R_x)T} + E \right]^{-1} \int_t^{t+T} e^{-(1/R_x)C(t-s)} \frac{1}{C} \varphi(u, s) \, ds, \\
y &= \left[ e^T + E \right]^{-1} \int_t^{t+T} e^{-(t-s)} \psi(s) \, ds. \tag{3.3}
\end{align*}
\]

According to [4] this solution can be considered as an action of the operators \( L_1, L_2 \) on the functions \( \varphi(u, t), \psi(t) \). In other words we have

\[
\begin{align*}
x &= L_1[\varphi(u, t)], \\
y &= L_2[\psi(t)]. \tag{3.4}
\end{align*}
\]

System (3.4) is an operator system which is equivalent to system (3.2) in the set of \( C^1 \) functions, continuous in \( \mu \). The operators \( L_1 \) and \( L_2 \) are linear and bounded and hence there exist positive constants \( \rho_1 \) and \( \rho_2 \), such that the following inequalities are satisfied:

\[
\begin{align*}
\|L_1 \varphi(u, t)\| &\leq \rho_1 \|\varphi(u, t)\|, \quad t \in [0, T], \\
\|L_2 \psi(t)\| &\leq \rho_2 \|\psi(t)\|. \tag{3.5}
\end{align*}
\]

If we go back to system (3.1), the following operator system can be obtained:

\[
\begin{align*}
x(t, u, \mu) &= L_1[\mu A(y) + B(u)], \\
y(t) &= L_2 \left[ \int_0^1 f(x) \, dt \right]. \tag{3.6}
\end{align*}
\]
Now, according to (3.5) and (3.6) we construct the system of Lyapunov’s majorizing equations in the certain domain $0 \leq x \leq 1$:

$$z(u, \mu) = \rho_1 \Phi(\beta, \mu),$$

$$\beta(\mu) = \rho_2 \Psi(x),$$

(3.7)

where $\Phi(\beta, \mu)$ and $\Psi(\mu)$ are Lyapunov majorants for the right-hand side functions of (3.6), and $\beta \geq |y|, \ z \geq |x|$. Therefore, according to the properties of Lyapunov’s majorizing equations [4,6] the following theorem has been proved.

**Theorem 3.** Suppose that system (3.7) in the domain $0 \leq x \leq 1$, has positive solutions $z(u, \mu)$ and $\beta(\mu)$ for $0 \leq \mu \leq \mu_*$ and $z(u, \mu_*) \leq 1$. Then system (3.1) has for $\mu \in [0, \mu_*]$ solutions $(x, y)$, which are unique in the set of $C^1$ functions. These solutions can be found by the following convergent simple iterations:

$$x_k = L_1 [\mu A(y_{k-1}) + B(u)], \quad k = 1, 2, \ldots,$$

$$y_k = L_2 \left[ \int_0^1 f(x_{k-1}) \, dt \right],$$

$$x_0 \equiv x(0), \quad y_0 \equiv y(0).$$

(3.8)

**Remark 3.** If we consider iterations (3.8) and fix $t = \tau$, $0 \leq \tau \leq 1$ and $\mu = \mu_0, \mu_0 \in [0, \mu_*]$ we obtain the dynamic rules of the perturbed CNN and it can be used as a dynamic transform of an initial state $v_{xij}(0)$ and given input $v_{uij}$ at any time $t$. For $t \to \infty$, if state variable $v_{xij}(t) = \text{const}$, the output $v_{yij}(t) = \{0, 1\}$ (Theorem 1).

**Remark 4.** Any stable system equilibrium point (Section 2.2) is a limit point of a set of trajectories of the corresponding system of differential equations (3.1). Therefore, the state space of a perturbed CNN can be partitioned into a set of basins of attraction centered at the stable system equilibrium points.

Generally speaking, we may construct a map from the initial state space $[0, 1]^{M \times N}$ into one of many distinct stable system equilibrium points, the output space $\{0, 1\}^{M \times N}$. This dynamical map is defined by (3.8), fixing $t = \tau$ and it depends on $\mu$, in other words we have

$$F_\mu: [0, 1]^{M \times N} \to \{0, 1\}^{M \times N}.$$  

It is clear that the small parameter $\mu$ affects the stability of CNN, in sense that different limit points and basins of attraction will be obtained for different values of $\mu \in [0, \mu_*]$.

**Remark 5.** The application of method of Lyapunov’s majorizing equations for finding the solutions of the perturbed CNN (3.1) is a new approach. The advantage is that we can estimate the interval of values of $\mu$ in which these solutions exist and to find them with the convergent simple iterations (3.8). This allows us to use standard computer implementation for the calculations.
In general, it is difficult if not impossible to predict the behavior of complex nonlinear dynamical systems, but our analysis shows that we can prove the existence and find the solutions of the perturbed CNN by using method of Lyapunov’s majorizing equations.

4. Example and computer implementation

Let us consider a two-dimensional grid with $3 \times 3$ neighborhood system as it is shown in Fig. 1.

Let us consider the following dynamical system as an example for perturbed CNN:

\[
C \dot{x} = -\frac{1}{R_x} + \mu A(y),
\]

\[
\dot{y} = -y + \int_0^1 f(x) \, dt \tag{4.1}
\]

with $B(u) = 0$, $C = 10^{-9}F$, $R_x = 10^3 \Omega$,

\[
A = \begin{bmatrix}
0 & p_1 & 0 \\
p_2 & 2 & p_2 \\
0 & p_1 & 0
\end{bmatrix}, \quad \mu = 0.1.
\]

For the initial conditions:

\[
y(0) =
\begin{array}{ccc}
0.8 & 0.7 & 1.0 \\
1.0 & 1.0 & 1.0 \\
-0.1 & 0.8 & -0.1
\end{array}
\]
we obtain the following state equilibrium points by using ODEX [3]:

\[
x(0) =
\begin{array}{ccc}
-0.9 & -1.0 & 1.0 \\
-1.0 & 1.0 & -1.0 \\
0.9 & 0.8 & 0.1 \\
\end{array}
\]

\[
y^* =
\begin{array}{ccc}
-0.56 & -0.2 & 0.61 \\
0.61 & 0.61 & 0.61 \\
0.25 & -0.56 & 0.25 \\
\end{array}
\]

\[
x^* =
\begin{array}{ccc}
-0.12 & -0.62 & 0.62 \\
-3.94 & 2.02 & -3.98 \\
0.12 & -0.56 & 0.62 \\
\end{array}
\]

The tolerance is \( \delta = 10^{-20} \).

**Remark 6.** The actual stable state equilibrium points attained by each cell clearly depends on its initial state (as well as those of its neighbor cells). But, even the initial conditions are very different their corresponding final states are virtual identical.

**References**